HomeWork 10

1 The Born approximation for the Yukawa potential

Let us consider the potential of the form:

$$V(r) = V_0 \frac{e^{-\alpha r}}{r}$$

where V_0 and α are real constants, with α positive. The potential is attractive or repulsive depending whether V_0 is negative or positive.

1.1 Plot V(r). What is the characteristic range of the interaction?

1.2 We assume $|V_0|$ to be small. Within the Born approximation, calculate the scattering amplitude $f_k(\theta, \phi)$.

1.3 Deduce from the result of last question that the differential cross section is given by:

$$\sigma(\theta) = \frac{4\mu^2 V_0^2}{\hbar^4} \frac{1}{(\alpha^2 + 4k^2 \sin^2 \theta/2)^2}$$

How do you explain that $\sigma(\theta)$ does not depend on the azimuthal angle ϕ ? Compare $\sigma(0)$ with $\sigma(\pi)$. Comment.

1.4 Calculate the total scattering cross section.

1.5 Examine the limit $\alpha \to 0$ of the preceding result. Comment.

2 Some basic properties of the density matrix

2.1 *Density matrix in quantum mechanics*

Another description of quantum mechanics can be obtained using the so-called density matrix approach, where the central object is now :

$$\rho(t) = |\Psi(t)\rangle\langle\Psi(t)|.$$

$\Psi(t)$ is the wave function of the system satisfying the Schrdinger equation.

a) Show that the evolution of ρ is ruled by the following equation:

$$\frac{d\rho}{dt} = -i[H, \rho]$$

H being the Hamiltonian of the system, as you guessed ...

b) Derive an expression for the mean value of an observable O, ($\langle\Psi|O|\Psi\rangle$).

c) Consider a beam of light travelling in the $z-$ direction. We define $|x\rangle$ and $|y\rangle$ to be respectively the $x-$ and $y-$ polarized states.

$$|x\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$|y\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Any pure state can be written as $|\Psi\rangle = a|x\rangle + b|y\rangle$. Write down the density matrix in the following cases :

- $x-$polarized state ($a = 1$, $b = 0$)
- $y-$polarized state ($a = 0$, $b = 1$)
- $45-$polarized state ($a = 1/\sqrt{2}$, $b = 1/\sqrt{2}$)
- $135-$polarized state ($a = -1/\sqrt{2}$, $b = 1/\sqrt{2}$)
- mixture of 50% $x-$polarized and 50% $y-$polarized states
- mixture of 50% $45-$polarized and 50% $135-$polarized states

2.2 *Density matrix in statistical physics*

Consider a system in thermodynamic equilibrium with a reservoir at temperature T. The density matrix is defined as:

$$\rho = Z^{-1} e^{-\beta H}$$

with $\beta = 1/k_B T$ as usual, H is the Hamiltonian of the system. Z is a normalization chosen so as to make the trace of ρ equal to 1.

$$Z = \mathrm{Tr} e^{-\beta H}$$

a) Show that in the basis of eigenvectors of H, ρ is diagonal.

b) From now on, we call $\rho_U = e^{-\beta H}$, the unnormalized density matrix. Prove that ρ_U satisfies the following differential equation:

$$-\frac{\partial \rho}{\partial \beta} = H\rho$$

with initial condition $\rho(0) = 1$.

We can write this equation in the position representation as follows:

$$-\frac{\partial \rho(x, x'; \beta)}{\partial \beta} = H(x)\rho(x, x'; \beta)$$

with initial condition

$$\rho(x, x'; 0) = \delta(x - x')$$

c) Let's solve the equation above for a simple example, the one-dimensional free particle. The Hamiltonian in this case is (if it is necessary to remind you !!!):

$$H = \frac{p^2}{2m}$$

Hint: you have to solve a diffusion equation - indications will be given during the tutorial ...

d) (BONUS) The same question, but now with an harmonic oscillator (a bit harder).